I- (3 points)

Let g be a function defined, on $]1\ ;+\infty[$, by: $g(x) = ax+\frac{b}{\ln x}$. (C) is the representative curve of g in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine a and b, knowing that (C) cuts the abscissa axis in a point E of abscissa e, and that the tangent to (C) at point E is parallel to the straight line (d) of equation: y = 2x.
2) In what follows, let a = 1 and b = –e.
   - a- Calculate the limits of g(x) at 1 and $+\infty$.
   - b- Set up the table of variation of g.
   - c- Show that the straight-line (D) of equation: y = x is an oblique asymptote to (C).
   - d- Draw (D) and (C).

II- (7 points)

Remark: The three parts of this question are independent.

Part A

In the complex plane refered to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points M and M′ of respective affixes z and z' such that $z' = \frac{2z}{1+z.\bar{z}}$.

Let z = x + iy and z′ = x′ + iy′, where x, y, x′, and y′ are real numbers.

1) Write x′ and y′ in terms of x and y.
2) Find the set of points M when $x' = \frac{1}{\sqrt{5}}$.

Part B

Given, in a direct orthonormal system $(O;\vec{u},\vec{v})$, the points A and B of repective affixes a = i and $b = -\sqrt{3}$. Let $Z = \frac{-a}{b-a}$.

1) Write Z in its algebraic form and its trigonometric form.
2) Give the geometric interpretation of | Z | and arg(Z).
3) What is the nature of triangle OAB?

Part C

Consider, in a direct orthonormal system $(O;\vec{u},\vec{v})$, the complex numbers $z_1 = 1 - i$ and $z_2 = -\sqrt{3}+i$.

1) Calculate the moduli and the arguments of z_1 and z_2.
2) Let $Z = \overline{z_1}\times(z_2)^n$, where n is a natural number. Write Z in trigonometric form.
3) In each of the following cases, find the natural number k.
   - a- $(z_1)^k$ is a real number.
   - b- $(z_1)^k$ is a pure imaginary number.
4) Deduce the nature of $(z_1)^{14}$.

III- (10 points)

Part A

Let g be a function defined, on $\mathbb{R}$, by: $g(x) = x^2 + (x - 1)e^x$.

1) Determine the limits of g(x) at $+\infty$ and at $-\infty$.
2) Show that $g'(x) = x(e^x + 2)$.
3) Set up the table of variations of g.
4) Prove that the equation g(x) = 0 admits, on $[0, +\infty[$, a unique solution α. Verify that $\frac{1}{2} < \alpha < 1$.

Part B

Consider the function f defined, on $[0, +\infty[$ by: $f(x) = \frac{e^x}{x + e^x}$. Let (C) be the representative curve of f in an orthonormal system $(O; \vec{i}, \vec{j})$.

1)
   - **a-** Show that $f(x) - x = \frac{-g(x)}{x + e^x}$.
   - **b-** Deduce that f(x) = x admits α as a unique solution.
2) Determine the limit of f(x) at $+\infty$. Interpret your result graphically.
3) Calculate $f'(x)$, then set up the table of varaitions of f.
4) Find the equation of (d), the tangent to (C) at the point of abscissa 0.
5) Draw (C) and (d).
6)
   - **a-** Show that f admits, on [0 , 1[, an inverse function h whose domain of definition is to be determined.
   - **b-** Does (C) and (H) have any point(s) in common? If yes, find its(their) coordinates.

GOOD WORK

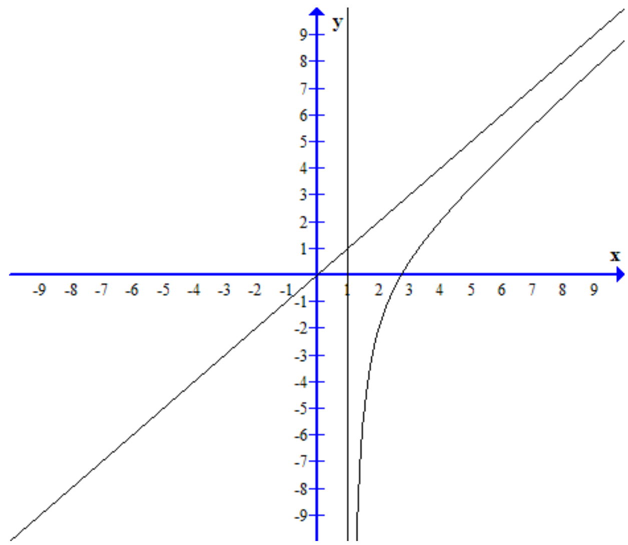
Mid-year Exam
Answer Key
Life Sciences

I.

1) g(e)=0 then $a\cdot e+\frac{b}{\ln e}=0$, then $a\cdot e+b=0$;

g´(e)=2 then $a+\frac{\frac{-b}{e}}{(\ln e)^2}=2$ then $a\cdot e-b=2e$.·

therefore, a = 1 and b = -e.



2)

a) $g(x)=x-\frac{e}{\ln x}$;

$\lim_{x\to 1^+} g(x) = 1$; then x = 1 V.A. $\lim_{x\to +\infty} g(x) = +\infty$.

b) $g'(x)=1+\frac{\frac{e}{x}}{(\ln x)^2}=1+\frac{e}{x(\ln x)^2}\succ 0$.

x	1	$+\infty$
g´(x)		+
g(x)		$+\infty$ ↗ $-\infty$

c) $\lim_{x\to+\infty}(g(x)-y_D)=0$ then y = x is an O.A.

II. **Part A**

1) $x'+iy'=\frac{2(x+iy)}{1+(x+iy)(x-iy)}=\frac{2x+2iy}{1+x^2+y^2}$.

$Re(z')=\frac{2x}{1+x^2+y^2}$; $Im(z')=\frac{2y}{1+x^2+y^2}$.

2) $Re(z')=\frac{2x}{1+x^2+y^2}=\frac{1}{\sqrt{5}}$; then $x^2+y^2-2x\sqrt{5}+1=0 \Rightarrow (x-\sqrt{5})^2+y^2=4$;

$M\in C(I;R)$ where $I(\sqrt{5};0)$ and $R=2$.

Part B

1) $Z=\frac{-i}{-\sqrt{3}-i}=\frac{1}{4}+\frac{\sqrt{3}}{4}i=\frac{1}{2}\ (\cos\frac{\pi}{3}\ +i\sin\frac{\pi}{3})$.

2) $|Z|=\frac{|Z_{\overrightarrow{OA}}|}{|Z_{\overrightarrow{AB}}|}=\frac{OA}{AB}$; $\arg(Z)=(\overrightarrow{AB},\overrightarrow{OA})$.

3) a is pure imaginary then A ∈ y′y. b is real then B ∈ x′x; hence $A\hat{O}B = \frac{\pi}{2}$ but $(\overrightarrow{AB}, \overrightarrow{AO}) = \frac{\pi}{3}$; then the triangle AOB is semi equilateral.

<u>Part C</u>

1) $|z_1| = \sqrt{2}$; arg(z_1) = $\frac{-\pi}{4}$; $|z_2| = 2$, arg(z_2) = $\frac{5\pi}{6}$.

2) $|Z| = |\overline{z_1}| \times |z_2|^n = \sqrt{2} \times 2^n = 2^n\sqrt{2}$; arg(Z) = arg($\overline{z_1}$) + arg$(z_2)^n$ = $(\frac{\pi}{4} + \frac{5n\pi}{6})$,

Thus, $Z = 2^n\sqrt{2}\left(\cos(\frac{\pi}{4} + \frac{5n\pi}{6}) + i\sin(\frac{\pi}{4} + \frac{5n\pi}{6})\right)$

3) $\arg(z_1{}^m) = \frac{-m\pi}{4}$;

a) $z_1{}^m$ ∈ IR then $\arg(z_1{}^m) = \frac{-m\pi}{4} = k\pi \Rightarrow m = 4k$; where k ∈Z, m ∈IN.

m ∈ {0,4,8,...}..

b) $z_1{}^m$ pure imaginary then $\arg(z_1{}^m) = \frac{-m\pi}{4} = \frac{(2k+1)\pi}{2}$; then m = -2(2k+1),

where k ∈Z, m ∈IN, so, m ∈ {2, 6, 10, 14, ...}.

c) m = 14, then $z_1{}^{14}$ is pure imaginary.

III. **<u>Part A</u>**

1) $\lim_{x\to+\infty} g(x) = +\infty$; $\lim_{x\to-\infty}(xe^x - e^x + x^2) = 0 + 0 + \infty = +\infty$

2) $g'(x) = e^x(x-1) + e^x + 2x = x(e^x + 2)$

3) g′(x) same sign as x

x	$-\infty$		0		$+\infty$
g′(x)		−	0	+	
g(x)	$+\infty$	↘	-1	↗	$+\infty$

4) Over [0; +∞[, g(x) is continuous is strictly monotonous (increasing) and changes its sign (-1 to +∞), then g(x) = 0 admits a unique solution α.

But $g(\frac{1}{2}) \times g(1) < 0$ then $\frac{1}{2} < \alpha < 1$.

Part B

1) $f(x) - x = \frac{e^x}{e^x + x} - x = \frac{-[xe^x - e^x + x^2]}{e^x + x} = \frac{-g(x)}{e^x + x}$.

$f(x) = x \Rightarrow f(x) - x = 0$, then $g(x) = 0$, but α is a unique solution of $g(x) = 0$ then α is a unique solution of $f(x) = x$.

2) $\lim_{x\to+\infty} f(x) \overset{HR}{=} \lim_{x\to+\infty} \frac{e^x}{e^x+1} \overset{HR}{=} \lim_{x\to+\infty} \frac{e^x}{e^x} = 1$; y = 1 H.A.

3) $f'(x) = \frac{e^x(e^x + x) - (e^x + 1)e^x}{(e^x+x)^2} = \frac{e^x(x-1)}{(e^x+x)^2}$. f′(x) has the same sign as (x-1)

x	0		1		$+\infty$
f′(x)		−	0	+	
f(x)	1	↘	$\frac{e}{e+1}$	↗	1

4) (d): y = f′(0)(x-0)+f(0)

(d): y = -x+1

5)

6)

a) f over [1; $+\infty$[is continuous and strictly monotonous (increasing), then it admits an inverse function f^{-1}; $D_{f^{-1}}$= f([1; $+\infty$[)= [$\frac{e}{e+1}$; 1 [.

b)