Class: General Sciences

Subject: Mathematics

Mid-Year Exam

I- (1 point)

In the table below only one, among the proposed answers to each question, is correct. Write down the number of each question and give, **with justification**, the corresponding answer.

Questions		Proposed Answers		
		A	**B**	**C**
1)	A and B are two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.45$. $P(\overline{A} \cap B) =$	0.05	0.15	0.25
2)	The coefficient of x^4y^{15} in the expansion of $(3x^2 - 2y^3)^7$ is:	–6048	6048	–21

II- (3 points)

Remark: The parts of this question are independent.

1) In an orthonormal system $(O\ ; \vec{i}\ , \vec{j})$, consider the points A(–2 , 0), B(1 , 0), and D(2 , 0). Let (C) be a variable circle that is tangent to the abscissa axis at point B. Let M be the common point between the tangents to (C) at points A and D.

Prove that M moves on a conic (H) whose equation is to be determined.

2) Let (C_m) be the family of curves of equation: $x^2 + my^2 - 2x - m^2 + 6m - 8 = 0$, where m is a real parameter. Study, according to the values of m, the nature of (C_m).

III- (3 points)

In an orthonormal system $(O; \vec{i}\ , \vec{j})$, consider the ellipse (E) of principal vertex A(1 , 0), directrix (D) of equation x = 3, and eccentricity $e = \frac{1}{2}$.

1) Specify the focal axis of (E) and verify that O is a focus of (E).

2) Prove that A′(–3 , 0) is the second principal vertex of (E), and deduce the center I of (E) and its second focus F.

3)

- **a-** Write an equation of (E).
- **b-** Specify the coordinates of the secondary vertices B and B′ of (E).

4) Let L be the common point of (E) with the ordinate axis (having a positive ordinate) and H be the common point of (D) with the abscissa axis. Prove that the straight-line (LH) is tangent to (E).

5) Calculate the area of the region limited by (E), the abscissa axis, the straight-line (LH), and the straight-line (d) of equation: x = –1.

IV- (3 points)

In a coeducational secondary school, the physician found that 60% of the students are males. Among those males, 20% wear eyeglasses. Three quarters of the females do not wear eyeglasses.

Part A

One student is chosen randomly from this school.

1) Calculate the probability that the chosen student is a male who wears eyeglasses.
2) Calculate the probability that the chosen student is a female given that she wears eyeglasses.

Part B

The director of the school wants to choose a group of **three students** randomly to represent the school abroad. He knows that the school has 250 students.

Consider the following events:

M_1: "The chosen group consists of one male and two females."
W: "The chosen students wear eyeglasses."

1) Construct a table that summarizes the above information.
2) Calculate the following probabilities: $P(M_1 / W)$ and $P(M_1 \cap W)$.
3) Let X be the random variable that designates the number of males in the chosen group.
   - a- Prove that $P(X = 1) = \dfrac{1485}{5146}$.
   - b- Find the probability distribution of X.
   - c- Prove that the expected value (mathematical mean) of X is 1.8 .
   - d- Find the distribution function of X and draw its representative curve (C).

V- (3 points)

Let (U_n) be a numerical sequence defined by: $U_0 = 0$ and $U_{n+1} = \dfrac{\sqrt{2}}{2}\sqrt{1+U_n}$, where n is a natural number.

Part A

1) Prove, by mathematical induction, that $\dfrac{\sqrt{2}}{2} \le U_n \le 1$, where $n \ge 1$.
2) Show, by mathematical induction, that (U_n) is an increasing sequence.
3) Deduce that the sequence (U_n) is convergent and determine its limit L.

Part B

1) Prove, for every real number x in $[0, \pi]$, that: $\sqrt{\dfrac{1+\cos x}{2}} = \cos\left(\dfrac{x}{2}\right)$.

   (Hint: You may use the formula: $\cos^2 \alpha = \dfrac{1+\cos 2\alpha}{2}$.)

2) Prove, by mathematical induction, that $U_n = \cos\left(\dfrac{\pi}{2^{n+1}}\right)$, where n is a natural number.
3) Find, again, the limit L of U_n.

VI- (7 points)

Part A

Let f be a function defined, on $[0\ ,+\infty[$, by: $f(x) = \dfrac{e^{x}+e^{-x}}{2}$. (C) is the representative curve of f in an orthonormal system $\left(O;\ \vec{i}\ ,\vec{j}\right)$.

1) Study the variations of f, then draw (C).
2) Consider the function h defined, on $[0\ ,+\infty[$, by: $h(x) = f(x) - x$.
   - **a-** Solve the equation: $e^{x} - e^{-x} - 2 = 0$.
   - **b-** Prove that h admits a minimum m that is strictly positive. Give an approximate value of m to the nearest 10^{-2}.
3) Consider the sequence (U_n) defined by: $U_0 = 1$ and $U_{n+1} = f(U_n)$, where n is a natural number.
   - **a-** Prove that $U_{n+1} - U_n$ is bounded below by m.
   - **b-** Deduce that $U_n \geq U_0 + n.m$, then deduce the limit of U_n.
4) Let a be a given real number. Discuss, **graphically**, the number of roots of the equation $f(x) = a$.

Part B

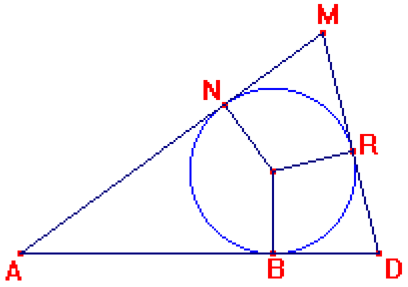
Let g be a function defined, on $[1\ ,+\infty[$, by: $g(x) = \ln\left(x+\sqrt{x^{2}-1}\right)$. (G) is the representative curve of g in the previous system $\left(O;\ \vec{i}\ ,\vec{j}\right)$.

1) Study the variations of g.
2) Let (D) be the straight-line of equation: $y = x$. Draw (G) and (D).
3) Sami states that (C) and (G) are symmetric with respect to straight-line (D). Is Sami right? Justify your answer.

GOOD WORK

Mid-year Exam
Answer Key
Second (Sciences)

Question I				Mark
1)	**B**	**2)**	**A**	**1each**

Question II			Mark
1)	MA – MD = (MN + NA) – (MR + RD) = MN + AB – MR – DB = 3 – 1 = 2 = constant. M moves on a branch of a hyperbola of foci A and D. $x^2-\frac{y^2}{3}=1$		**3**
2)	• If m = 0: $(x-1)^2-9=0$; $(C_0)=(D_1)\cup(D_2)$, where (D_1): x = 4 and (D_2): x = –2 • If m = 1: $(x-1)^2+y^2=4$; (C_1) is a circle • If m ≠ 0 and m ≠ 1: $(x-1)^2+my^2=(m-3)^2$ ▪ If m = 3: $(x-1)^2+3y^2=0$; (C_3) is the point (1 , 0) ▪ If m ≠ 3: $\frac{(x-1)^2}{(m-3)^2}+\frac{y^2}{\frac{(m-3)^2}{m}}=1$ ○ If m < 0: (C_m) is a hyperbola ○ If m > 0 and m ≠ 0 and m ≠ 1 and m ≠ 3: (C_m) is an ellipse		**3**

Question III			Mark
1)		• The focal axis is the abscissa axis since the directrix is vertical and A belongs to the abscissa axis • $\frac{AO}{d(A,(D))}=\frac{1}{2}=e$, then O is a focus of (E)	**0.5** **0.5**
2)		• $\frac{A'O}{d(A',(D))}=\frac{1}{2}=e$, then A′ ∈ (E); but A′ ∈ (F.A.), then A′ is the second vertex of (E) • I midpoint of [AA′], then I(–1 , 0) • I midpoint of [OF], then F(–2 , 0)	**0.5** **0.5** **0.5**
3)	**a-**	$\frac{(x+1)^2}{4}+\frac{y^2}{3}=1$	**1**
	b-	$B\left(-1,\sqrt{3}\right)$ and $B'\left(-1,-\sqrt{3}\right)$	**0.25** **0.25**
4)		$L\left(0,\frac{3}{2}\right)$; H(3 , 0); Slope (LH) = $-\frac{1}{2}$ = y'_L and H ∈ (E)	**1**
5)		Area = Area of triangle – $\frac{1}{4}\pi ab=4-\frac{\pi\sqrt{3}}{2}\approx 1.28u^2$	**1**

Question IV	Mark
Part A	

<table>
<tr><td>1)</td><td colspan="2">$P(M \cap W) = 0.12$</td><td>0.5</td></tr>
<tr><td>2)</td><td colspan="2">$P(F / W) = \frac{P(F \cap W)}{P(W)} = \frac{0.1}{0.12 + 0.1} = \frac{5}{11}$</td><td>1</td></tr>
<tr><td colspan="4"><u>Part B</u></td></tr>
<tr><td>1)</td><td colspan="2"><table>
<tr><td></td><td>M</td><td>F</td><td></td></tr>
<tr><td>W</td><td>30</td><td>25</td><td>55</td></tr>
<tr><td>$\overline{W}$</td><td>120</td><td>75</td><td>195</td></tr>
<tr><td></td><td>150</td><td>100</td><td>250</td></tr>
</table></td><td>0.5</td></tr>
<tr><td>2)</td><td colspan="2">$P(M_1 / W) = \frac{C_{30}^{1} \times C_{25}^{2}}{C_{55}^{3}} = \frac{200}{583}$; $P(M_1 \cap W) = P(M_1 / W) \times P(W) = \frac{C_{30}^{1} \times C_{25}^{2}}{C_{250}^{3}} = \frac{9}{2573}$</td><td>0.5
0.5</td></tr>
<tr><td rowspan="4">3)</td><td>a-</td><td>$P(X = 1) = P(1M \text{ and } 2F) = \frac{C_{150}^{1} \times C_{100}^{2}}{C_{250}^{3}} = \frac{1485}{5146}$</td><td>0.5</td></tr>
<tr><td>b-</td><td>$X_\Omega = \{0 , 1 , 2 , 3\}$<table>
<tr><td>$X = x_i$</td><td>0</td><td>1</td><td>2</td><td>3</td><td>Total</td></tr>
<tr><td>$P(X = x_i)$</td><td>$\frac{1617}{25730}$</td><td>$\frac{1485}{5146}$</td><td>$\frac{2235}{5146}$</td><td>$\frac{5513}{25730}$</td><td>1</td></tr>
</table></td><td>1</td></tr>
<tr><td>c-</td><td>$E(X) = 1.8$</td><td>0.5</td></tr>
<tr><td>d-</td><td>Easy</td><td>1</td></tr>
</table>

<table>
<tr><td colspan="2">Question V</td><td>Mark</td></tr>
<tr><td colspan="3"><u>Part A</u></td></tr>
<tr><td>1)</td><td>Easy</td><td>1</td></tr>
<tr><td>2)</td><td>Easy</td><td>1</td></tr>
<tr><td>3)</td><td>• (U_n) is convergent being an increasing sequence that is bounded from above by 1
• (U_n) is convergent
(U_n) is defined recursively
$f(x) = \frac{\sqrt{2}}{2}\sqrt{1 + x}$ is continuous
then $L = \lim\limits_{n \to +\infty} U_n$ is a root of the equation $L = f(L)$; $L = 1$ or $L = -0.5$ (rejected since (U_n) is an increasing sequence whose first term is zero)</td><td>1</td></tr>
<tr><td colspan="3"><u>Part B</u></td></tr>
<tr><td>1)</td><td>$\cos^2\left(\frac{x}{2}\right) = \frac{1 + \cos x}{2}$; $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$; $x \in [0 , \pi]$, then $\frac{x}{2} \in \left[0 , \frac{\pi}{2}\right]$, then $\cos\left(\frac{x}{2}\right) \geq 0$, then $\cos\left(\frac{x}{2}\right) = +\sqrt{\frac{1 + \cos x}{2}}$</td><td>1</td></tr>
<tr><td>2)</td><td>Easy</td><td>1</td></tr>
<tr><td>3)</td><td>$L = \lim\limits_{n \to +\infty} U_n = \lim\limits_{n \to +\infty} \cos\left(\frac{\pi}{2^{n+1}}\right) = 1$</td><td>1</td></tr>
</table>

<table>
<tr><th colspan="4">Question VI</th><th>Mark</th></tr>
<tr><td colspan="5"><u>Part A</u></td></tr>
<tr><td>1)</td><td colspan="2">$\begin{array}{c|ccc} x & 0 & & +\infty \\ \hline f'(x) & & + & \\ \hline f(x) & 1 & \nearrow & +\infty \end{array}$</td><td>Curve</td><td>2
1</td></tr>
<tr><td rowspan="2">2)</td><td>a-</td><td colspan="2">$x = \ln\left(1+\sqrt{2}\right)$</td><td>1</td></tr>
<tr><td>b-</td><td colspan="2">• At x = = $\ln\left(1+\sqrt{2}\right)$: $h'\left(\ln\left(1+\sqrt{2}\right)\right)$ = 0 and h′(x) changes signs (– to +); then h has a minimum m
• $h\left(\ln\left(1+\sqrt{2}\right)\right) \approx 3.22$</td><td>1
0.5</td></tr>
<tr><td rowspan="2">3)</td><td>a-</td><td colspan="2">$U_{n+1} - U_n = f(U_n) - U_n = h(U_n) \geq m$</td><td>0.5</td></tr>
<tr><td>b-</td><td colspan="2">• $U_1 - U_0 \geq m$; $U_2 - U_1 \geq m$; … ; $U_n - U_{n-1} \geq m$; (+) $U_n - U_0 \geq m + m + \ldots + m = nm$; $U_n \geq U_0 + nm$
• $\lim\limits_{n\to+\infty} U_n \geq +\infty$; then $\lim\limits_{n\to+\infty} U_n = +\infty$</td><td>1
0.5</td></tr>
<tr><td>4)</td><td colspan="3">a < 1: no roots; a ≥ 1: one root</td><td>1</td></tr>
<tr><td colspan="5"><u>Part B</u></td></tr>
<tr><td>1)</td><td colspan="3">Easy</td><td>2</td></tr>
<tr><td>2)</td><td colspan="3">Easy</td><td>1.5</td></tr>
<tr><td>3)</td><td colspan="3">$y = \ln\left(x+\sqrt{x^2-1}\right)$; $e^y = x + \sqrt{x^2-1}$; $e^y - x = \sqrt{x^2-1}$; $e^{2y} + x^2 - 2xe^y = x^2 - 1$; $x = \frac{e^y + e^{-y}}{2}$; $g^{-1}(x) = \frac{e^x + e^{-x}}{2} = f(x)$; Sami is right</td><td>2</td></tr>
</table>