صف العلوم العامة الانكليزي المادة رياضيات

Midyear exam

I- (2.5points)

In the following table only one of the proposed answers to each question is correct, choose the correct answer and justify.

No	Question	Answer		
		a	b	c
1	Given the points : A(2,1,-1),B(3,0,1), C(2,-1,3)& D(0,0,a),one of the values of a so that the volume of ABCD =5 u^3 is:	$a=-16$	$a=16$	$a=14$
2	The equation of Ellipse of focal vertices $A(4,0)$ and $A'(-2,0)$, and the distance between the directrices equal $=\frac{18}{\sqrt{5}}$	$4x^2+9y^2-8x-32=0$	$\frac{(x-1)^2}{9}+y^2=1$	$\frac{(x-1)^2}{9}+\frac{y^2}{5}=1$
3	$\lim_{x\to-\infty}[\ln(e^{-2x}-2e^{-x}+1)+2x]=$	$-\infty$	$+\infty$	0
4	The equation of plane (Q) containing st line $(d): x=m-1, y=2m+1,$ $z=-2m+1$ and perpendicular to plane(p)2x+y+2z-1=0, is:	2x+2y-z+4=0	2x-2y-z+4=0	2x-2y-z+5=0

II- (3 points)

In the space of orthonormal system $(o,\vec{i},\vec{j},\vec{k})$, consider the point A (0 ,-1 , 3) and the st. line (d) of equation: $x=t, y=-t+1, z=1$.

1) Show that : $x+y+z-2=0$ is the equation of the plane (P) determined by point A and st. line (d).
2) Find the equation of plane (Q) passing through A and perpendicular to (d), the deduce the coordinates of point H the orthogonal projection of A on (d).
3) Consider the circle(C) in plane (p) of center A and tangent to (d) .
a) Calculate the radius of (C).
b) Show that B (-1,-2, 5) is a point of circle (C) diametrically opposite to H, and write the equation of the tangent to circle (C) at B.
4) i) Show that the st. line of (Δ):$x=\lambda, y=\lambda-1, z=\lambda+3$, is the axis of circle (C) .
   ii) Point E ($\sqrt{6},\sqrt{6}-1,\sqrt{6}+3$) belong to (Δ).
   Show that HBE is equilateral triangle and calculate its area.

III- (3 points)

In the plane of orthonormal system $x'ox, y'oy$, consider the s. line (d) of equation $x = 6$

And the midpoint M such that $\frac{OM}{MH} = \frac{1}{2}$, where H is the orthogonal projection of M on (d).

1) Show that the equation of (C) the set of M is written in the form:

   $3x^2 + 4y^2 + 12x - 36 = 0$

2) Find the vertices of the focal axis, the vertices of the non focal axis, the other focus and draw (C).
3) E is the point of intersection of (C) with the $y - axis$ of positive ordinate.

   Find the equation of (T) the tangent to (C) at E.

4)
   a) Show that $I(\frac{6}{5}, \frac{12}{5})$ is the orthogonal projection of the point O on (T)
   b) Show that I belongs to the auxiliary circle of (C).

IV- (6points)

Consider the function g defined over $[-1, +\infty[$ by : $g(x) = \sqrt{x+1} - 1$ where (P_1) is its representative curve in an orthonormal system $(O, \vec{i}, \vec{j})$

Part A:

1. Calculate $\lim_{x \to +\infty} g(x)$, $\lim_{x \to +\infty} \frac{g(x)}{x}$, interpret.
2. Prove that g is increasing over $[-1, +\infty[$ and setup the table of variation of g.
3. Calculate $g(0)$, and deduce the sign of $g(x)$ over $[-1, +\infty[$
4. Draw (P_1).
5. Let $I = [0, 3]$.
   a. Prove that $g(I)$ is included in I.
   b. Prove that $|g'(x)| \leq \frac{1}{2}$ for every $x \in I$, and prove that $|g(x)| \leq \frac{1}{2}|x|$.
   c. Consider the sequence (U_n) that is defined by: $U_0 = 1$ and for every $n \in \mathbb{N}$, $U_{n+1} = g(U_n)$.
      i- Prove by mathematical induction that U_n belongs to I for every n.
      ii- Prove that $|U_{n+1}| \leq \frac{1}{2}|U_n|$.
      iii- Prove that $|U_n| \leq \frac{1}{2^n}$ and deduce the limit of U_n as n tends to $+\infty$.

PART B: For every point $M(x, g(x))$ on (P_1) consider its symmetric $M'(x', y')$ with respect to the line of equation $y = -1$. The set of point of M' is the curve (P_2) , which is the representative curve of the function: $h(x) = -\sqrt{x+1} - 1$.

i. Draw (P_2) on the same system.

ii. Prove that $(P) = (P_1) \cup (P_2)$ is the representative curve of the set of points whose coordinates satisfy the expression : $y^2 + 2y - x = 0$

iii. Prove that this expression represents the equation of a parabola whose elements are to be determined.

iv. Find the equation of the normal to (P) at the point of ordinate 0.

PARTC: Let f be the function defined by $f(x) = \ln(g(x))$ where (C) is its representative curve in an orthonormal system.

1) Show that the domain of f is:$]0, +\infty[$.

2) Find the limits of f at the end points of the domain.

3) Study the variations of f and sketch its table of variation.

4) Solve the equations: $f(x) = 0$ &$f(x) = -ln2$, and draw (C).

V- ***(5.5 points)***

Consider the function f defined on $\mathbb{R} =]-\infty, +\infty[$ $by : f(x) = \frac{e^x}{e^x - x}$.

PART A:

1) Let g be the function defined over $]-\infty, +\infty[$ by $g(x) = e^x - x - 1$

a- Find $g'(x)$, sketch the table of variations of g.Deduce that: $g(x) > 0$.

b- Verify that the expression $\frac{e^x}{e^x - x}$ is defined over: $\mathbb{R}$.

2)

a- Find the $\lim_{x \to -\infty} f(x)$& $\lim_{x \to +\infty} f(x)$. Interpret. Deduce the asymptotes of (C).

b- Show that : $f'(x) = \frac{(1-x)e^x}{(e^x - x)^2}$ and draw the table of variations of f.

c- Draw (C).

3) Let (T) be the representative curve of h defined over $\mathbb{R}$ by $h(x) = \frac{1}{e^x - x}$.

i) Show that the axis of abscissa is the asymptote of (T) at the ends of its domain.

ii) Find $h'(x)$, sketch the table of variations of h and draw (T) in the same system of$f$.

iii) Study according to the values of the relative position of the curve (C) W.r.t. (T) .

iv) Calculate the area of region bounded by (C) ,(T) and x=1.

PART B :

The study of the sequence: $\boldsymbol{U_n = \int_0^n [f(x)]dx}$ and the sequence $\boldsymbol{V_n = U_n - n}$.

a. Verify that: $U_{n+1} - U_n = \int_n^{n+1} f(x)dx$, deduce sense of variations of U_n.

b. Verify that $f(x) = 1 + \frac{x}{e^x - x}$, and show that: $U_n = n + \int_0^n \frac{x}{e^x - x} dx$.

c. Prove that V_n is increasing.

d. Assume it is given that for all $x \in]0,+\infty[$ we have: $e^x - x \geq \frac{e^x}{2}$

deduce that: $0 < V_n \leq \int_0^n 2xe^{-x}dx$.

e. Calculate: $\int_0^n 2xe^{-x}dx$, and deduce that: $0 < \lim_{n\to+\infty} V_n \leq 2$.

GOOD LUCK ☺

Correction of midyear Exam

I- (5points/40)

1- $\overrightarrow{BC}.(\overrightarrow{BA}\wedge\overrightarrow{BD}) = \begin{vmatrix} -1 & -1 & 2 \\ -1 & 1 & -2 \\ -3 & 0 & a-1 \end{vmatrix}$

$= -1(a-1) + 1[(-a+1) - 6] + 2(3)$

$= -a + 1 - a - 5 + 6$

$= -2a + 2$

$Volume = \frac{|-2a+2|}{6} = 5\,; |-a+1| = 15$

$-a = 14 \;;\; a = -14$

$-a + 1 = -15 \;;\; a = 16$ Correct: B (1.5pt)

2- $4x^2 + 9y^2 - 8x - 32 = 0$ Correct: A (1.5pt)

3- $\lim_{x\to-\infty}[\ln(e^{-2x} - 2e^{-x} + 1) + 2x]$

$=\lim_{x\to-\infty}[\ln(e^{-2x} - 2e^{-x} + 1) + \ln e^{2x}]$

$=\lim_{x\to-\infty}[\ln(e^{-2x} - 2e^{-x} + 1)e^{2x}]$

$=\lim_{x\to-\infty}[\ln(1 - 2e^{x} + e^{2x})] = \ln 1 = 0$ Correct C (1pt)

4- 2x-2y-z+5=0

Correct C (1pt0

II- (6pts/40)

1) $A \notin (d)$ since $x_A = 0$, $t = 0$, $y_A = -1$, $t = 2$

$A \in (p)$ since $0 - 1 + 3 - 2 = 0$

(d) lies in (p) since $t + (-t+1) + 1 - 2 = 0$ $0 = 0$

2) $N_Q = \overline{V_d} = (1, -1, 0)$

$Q: x - y + r = 0$ $(A \in Q)$

$0 + 1 + r = 0$ $r = -1$

$Q: x - y - 1 = 0$; $H \in (d)$ since $t = 1$

$H \in Q$ since $1 - 0 - 1 = 0$

Or $(H) \in (d)$ and $\overline{AH}.\overline{V_d} = 0$

3)

a- Radius of (C) is $AH = \sqrt{1+1+4} = \sqrt{6}$

b- $x_A = \frac{x_B + x_H}{2}$, $0 = \frac{-1+1}{2}$, $y_A = \frac{y_b + y_h}{2} = \frac{-2+0}{2} = -1$, $z_A = 3$ Then (B) is diametrically opposite to "H"

Equation of tangent $(T) to (C)$ at B:(T) is parallel to $(d) x = m + (-1) y = -m - 2$, $z = 5$

4) $i)$ (Δ) perpendicular on (P) ; $\overline{V_\Delta} = \overline{N_p}$ $(1, 1, 1)$ Then Δ: $\begin{cases} x = \lambda \\ y = \lambda - 1 \\ z = \lambda + 3 \end{cases}$

$ii)$ $\lambda = \sqrt{6}$;then B belong to Δ,$HB = BE = HE$ $area: \frac{EI \times HB}{2} = \frac{\frac{HB \times \sqrt{3}}{2} \times HB}{2} = \frac{\sqrt{432}}{2}$unit area.

III- (6pts/40)

1) $H(6, y) \epsilon (d)$, $2(OM)^2 = (MH)^2$; $4(x^2 + y^2) = (x - 6)^2 + (y - y)^2$

Then $3x^2 + 4y^2 + 12x - 36 = 0$ is (C) set of M.

2) $(C): \frac{(x+2)^2}{16} + \frac{y^2}{12} = 1$ $a = 4$ $b = 2\sqrt{3}$ center $W(-2, 0)$

Ellipse

In $X'wX, Y'wY$	$in\ x'ox, y'oy$
$A(4,0)$ $A'(-4,0)$	$A(2,0)$ $A'(-6,0)$
$B(0, 2\sqrt{3})$ $B'(0, -2\sqrt{3})$	$B(-2, 2\sqrt{3})$ $B'(-2, -2\sqrt{3})$
$F(2,0)$ $F'(-2,0)$	$F(0,0)$ $F'(-4,0)$

3) $3x^2 + 4y^2 + 12x - 36 = 0$

$6x + 8yy' + 12 = 0$

$x = 0$ $4y^2 - 36 = 0$

$y^2 = 9$ then $y = \pm 3$

$(0, 3)$ then $y'_{(o)} = \frac{-12}{8(3)} = -\frac{1}{2}$

then equation of tangent $y = -\frac{1}{2}x + 3$

4)

a) $I(\frac{6}{5}, \frac{12}{5})$ is a point on (T) since $\frac{12}{5} = -\frac{6}{10} + 3$

Slope of OI: $\frac{\frac{12}{5}}{\frac{6}{5}} = 2$ slope of (T)=$-\frac{1}{2}$ then OI perpendicular on (T)

b) $I \in$ to the auxiliary circle $R = a = 4$ center $w(-2, 0)$ of equation $(x+2)^2 + y^2 = 16$

Then coordinates of I verify the equation of the circle $I(\frac{6}{5}, \frac{12}{5})$ then $\left(\frac{6}{5}+2\right)^2 + \left(\frac{12}{5}\right)^2 = 16$

or prove that $WI = a = 4$.

IV-(13point)

Part A:

1) $\lim_{x\to+\infty} g(x) = +\infty$ $\lim_{x\to+\infty} \frac{g(x)}{x} = \lim_{x\to+\infty} \frac{\sqrt{x+1}-1}{x} = 0$ Asymptotic direction parallel to x'x

2) $g'(x) = \frac{1}{\sqrt{x+1}} > 0$; $g(-1) = -1$

x	-1		$+\infty$
g'		$-$	
g	$+1$	$\nearrow$	$+\infty$

3) $g(0) = 0$, $g(x) < 0$ when $1- \leq x < 0$, $g(x) > 0$ when $x > 0$

4) Graph

5)

a) $g(0) = 0$; $g(3) = 1$

$g([0,3]) = [0,1] \subset I$

b) $|g'(x)| - \frac{1}{2} = \frac{1}{\sqrt{x+1}} - 1 = \frac{1-\sqrt{x+1}}{\sqrt{x+1}} = \frac{(1-\sqrt{x+1})(1+\sqrt{x+1})}{\sqrt{x+1}(1+\sqrt{x+1})} = \frac{-x}{\sqrt{x+1}(1+\sqrt{x+1})} < 0$

$|g'(x)| \leq \frac{1}{2}$ for $x \in [0,3]$, $|g(b) - g(a)| \leq k|x-a|$, $|g(x)| \leq \frac{1}{2}|x|$

c)

i. $U_0 = 1 \in [0,3] = I$, $U_n \in [0,3]$

$0 < U_n < 3$ Then $g(0) < g(U_n) < g(3)$, $0 < U_{n+1} < 1$ then $U_{n+1} \in I$

ii. Since $|g(x)| < \frac{1}{2}|x|$ then $|g(U_n)| < \frac{1}{2}|U_n|$ then $|U_{n+1}| \leq \frac{1}{2}|U_n|$

iii. By mathematical induction $U_0 = 1 \leq \frac{1}{2^0} = 1$

Given $|U_n| \leq \frac{1}{2^n}$ show that $|U_{n+1}| \leq \frac{1}{2^{n+1}}$

Proof: $|U_{n+1}| \leq \frac{1}{2}|U_n| \leq \frac{1}{2} \times \frac{1}{2^n}$, $|U_{n+1}| \leq \frac{1}{2^{n+1}}$, $\lim_{n\to+\infty} U_n \leq \lim_{n\to+\infty} \frac{1}{2^n} = 0$

And $U_n \geq 0$ *then* $\lim_{n\to+\infty} U_n = 0$

PART B:

i) Figure is deduced by symmetry about $y = -1$

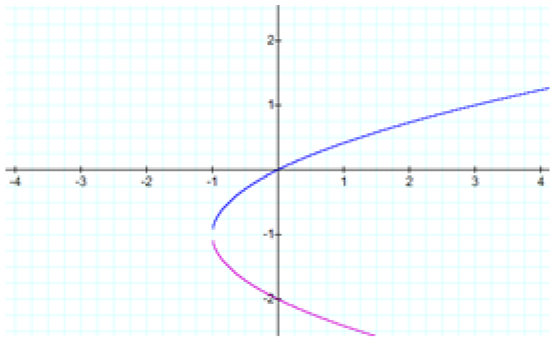
ii) $y = \pm\sqrt{x+1} - 1$; $y + 1 = \pm\sqrt{x+1}$

$y^2 + 2y + 1 = x + 1$ then $y^2 + 2y - x = 0$

iii) Parabola since $(y+1)^2 = x + 1$ then $= Y^2$, find vertex, directrix, parameter.

iv) Equation of tangent at o, $2yy' + 2y' - 1 = 0$, $y'_{(o)} = \frac{1}{2}$

Slope of normal $= -2$ equation of normal: $y = -2x$



Graph of $g(x) \cup h(x)$= $\pm\sqrt{\boldsymbol{x+1}} - \mathbf{1}$

PARTC:

1- $g(x) > 0$ for $x > 0$ then domain of $f(x)$ is $]0, +\infty[$

2- $\lim_{x\to 0} f(x) = \ln(f(0)) = \ln 0 = -\infty$

$\lim_{x\to+\infty} f(x) = \ln(+\infty) = +\infty$

$\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty$

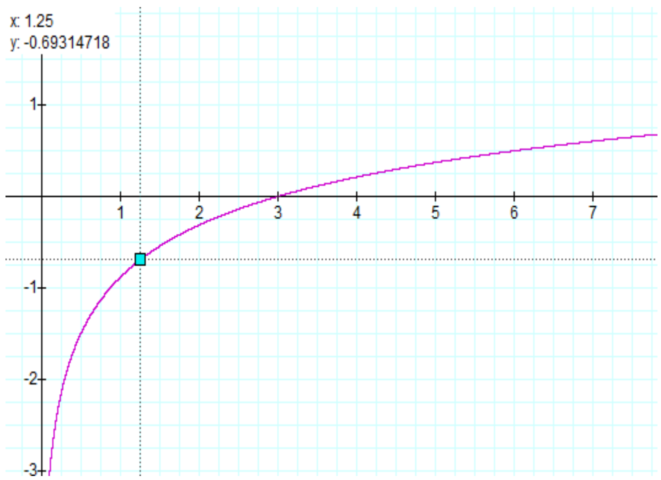
3- $f'(x) = \frac{g'(x)}{g(x)} > 0$ $g'(x) > 0$ and $g'(x) > 0$, $x \in]0, +\infty[$

x	0		$+\infty$
f'		$+$	
f	$-\infty$	$\nearrow$	$+\infty$

4- $f(x) = 0$ $g(x) = 1$ $x = 3$: (3, 0)

$f(x) = -\ln 2$ $\ln(g(x)) = -\ln 2 = \ln\frac{1}{2}$ Then $g(x) = \frac{1}{2}, \sqrt{x+1} - 1 = \frac{1}{2}$ then $x = \frac{5}{4}$

: $(\frac{5}{4}, -\ln 2)$



Graph of $f(x) = ln(g(x)) = ln(\sqrt{x+1} - 1)$

V- (10pts/40)

Part A

1)

a- $g'(x) = e^x - 1$

x	$-\infty$		0		$-\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	↘	0	↗	$-\infty$

$g(x) \geq 0$

b- $e^x - x - 1 \geq 0$, $e^x - x \geq 1$; $\frac{e^x}{e^x - x}$ is defined on $\mathbb{R}$

2) $f(x) = \frac{e^x}{e^x - x}$

a- $\lim_{x\to-\infty} f(x) = \frac{0}{+\infty} = 0$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x}{e^x\left(1-\frac{x}{e^x}\right)} = \lim_{x\to+\infty} \frac{1}{1-\frac{x}{e^x}} = \frac{1}{1-0} = 1$

$y = 0\ (H.A)\quad y = 1\ (H.A)$

b- $f'(x) = \frac{e^x(e^x - x) - (e^x - 1)e^x}{(e^x - x)^2} = \frac{e^x(-x+1)}{(e^x - x)^2}$

x	$-\infty$		1		$-\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0	↗	$\frac{e}{e-1}$	↘	1

c- Graph

3- a) $\lim_{x\to-\infty} h(x) = 0$, $\lim_{x\to+\infty} h(x) = o$, $y = 0\ (H.A)$.

b) $h'(x) == \frac{-e^x+1}{(e^x-x)^2}$

x	$-\infty$		0		$-\infty$
$h'(x)$		+	0	−	
$h(x)$	0	↗	1	↘	0

Graph (T)

c) $f(x) - h(x) = \frac{e^x-1}{e^x-x}$, for $x > 0$,(C) is below (T) & for $x < 0$,(C) is above (T)

Comon point (0,1)

d) Area=$\int_0^1 \frac{e^x-1}{e^x-x}dx = \ln(e^x - x)\,{}_0^1$=0.54 unit area.

PART B:

a. $U_{n+1} - U_n = \int_0^{n+1} f(x)dx - \int_0^n f(x)dx = \int_n^0 f(x)dx + \int_0^{n+1} f(x)dx$

$= \int_n^{n+1} f(x)dx > o$ and $(U_n)$is increasing.

b. $f(x) = 1 - \frac{e^x}{e^x+x}$, $U_n = \int_0^n dx + \int_0^n \frac{x}{e^x-x}dx = [x]_0^n + \int_0^n \frac{x}{e^x-x}dx = n + \int_0^n \frac{x}{e^x-x}dx$

c. $V_n = U_n - n + \int_0^n \frac{x}{e^x-x}dx - n = \int_0^n \frac{x}{e^x-x}dx$

$$V_{n+1} - V_n = \int_0^{n+1} \frac{x}{e^x - x}dx - \int_0^n \frac{x}{e^x - x}dx = \int_n^{n+1} \frac{x}{e^x - x}dx > 0$$

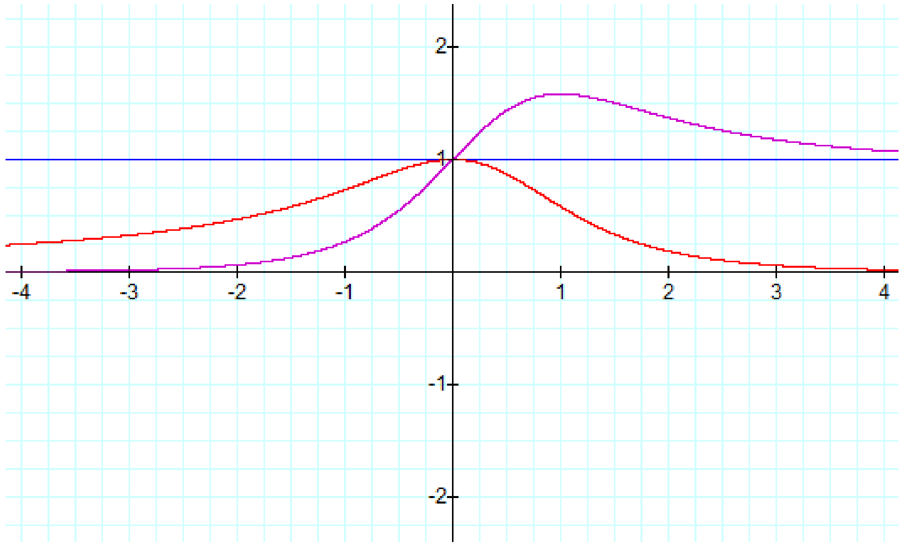
d. $e^x - x - \frac{e^x}{2} > 0$, $e^x - x > \frac{e^x}{2}$, $\frac{1}{e^x-x} < \frac{2}{e^x}$ and then $o < \frac{1}{e^x-x} < \frac{2}{e^x}$

x> 0 then $o < \frac{x}{e^x-x} < \frac{2x}{e^x}$

$0 < \int_0^n \frac{x}{e^x-x}dx \leq \int_0^n 2xe^{-x}$; $0 < \int_0^n \frac{x}{e^x-x}dx \leq \int_0^n 2xe^{-x}dx$

Then; $0 < V_n \leq \int_0^n 2xe^{-x}dx = \frac{2n-2}{e^n} + 2$

Then; $0 < \lim_{+\infty} V_n \leq \lim_{+\infty} \frac{2n-2}{e^n} + 2$; $0 < \lim_{+\infty} V_n \leq 2$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